

DIVISIBILITY OF WEIL SUMS OF BINOMIALS

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ABSTRACT. Consider the Weil sum $W_{F,d}(u) = \sum_{x \in F} \psi(x^d + ux)$, where F is a finite field of characteristic p , ψ is the canonical additive character of F , d is coprime to $|F^*|$, and $u \in F^*$. We say that $W_{F,d}(u)$ is three-valued when it assumes precisely three distinct values as u runs through F^* : this is the minimum number of distinct values in the nondegenerate case, and three-valued $W_{F,d}$ are rare and desirable. When $W_{F,d}$ is three-valued, we give a lower bound on the p -adic valuation of the values. This enables us to prove the characteristic 3 case of a 1976 conjecture of Hellese: when $p = 3$ and $[F : \mathbb{F}_3]$ is a power of 2, we show that $W_{F,d}$ cannot be three-valued.

1. INTRODUCTION

In this paper, we are concerned with Weil sums of binomials of the form

$$(1) \quad W_{F,d}(u) = \sum_{x \in F} \psi(x^d + ux),$$

where F is a finite field of characteristic p , the exponent d is a positive integer such that $\gcd(d, |F^*|) = 1$, the coefficient u is in F^* , and $\psi : F \rightarrow \mathbb{C}$ is the canonical additive character $\psi(x) = e^{2\pi i \operatorname{Tr}_{F/\mathbb{F}_p}(x)/p}$. Nontrivial Weil sums of form

$$\sum_{x \in F} \psi(ax^m + bx^n),$$

with $\gcd(m, |F^*|) = \gcd(n, |F^*|) = 1$ can be reparameterized to the form (1). Such sums and their relatives arise often in number-theory [17, 21, 10, 14, 4, 5, 18, 16, 9, 7, 8], and in applications to finite geometry, digital sequence design, error-correcting codes, and cryptography. See [15, Appendix] on the various guises in which these sums appear in these applications, and for a bibliography.

We fix F and d , and consider the values $W_{F,d}(u)$ attains as u varies over F^* , but typically ignore the trivial $W_{F,d}(0) = 0$, which is the Weil sum of a monomial. We say that $W_{F,d}$ is v -valued to mean that $|\{W_{F,d}(u) : u \in F^*\}| = v$.

If F is of characteristic p and d is a power of p modulo $|F^*|$, then $\psi(x^d) = \psi(x)$, so $W_{F,d}(u)$ effectively becomes the Weil sum of the monomial $(1+u)x$,

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so that

$$(2) \quad W_{F,d}(u) = \begin{cases} |F| & \text{if } u = -1, \\ 0 & \text{otherwise,} \end{cases}$$

and in this case we say that d is *degenerate over F* . For nondegenerate d , Helleseeth [13, Theorem 4.1] showed that one obtains more than two values.

Theorem 1.1 (Helleseeth, 1976). *If d is nondegenerate over F , then $W_{F,d}$ is at least three-valued.*

Much interest has focused on which choices of F and d make $W_{F,d}$ precisely three-valued, and ten infinite families have been found (see [1, Table 1]). From these, one finds that if F is of characteristic p , and if $[F : \mathbb{F}_p]$ is divisible by an odd prime, then there is a d such $W_{F,d}$ is three-valued. However, no three-valued examples have ever been found for fields F where $[F : \mathbb{F}_p]$ is a power of 2. This prompted the following conjecture [13, Conjecture 5.2].

Conjecture 1.2 (Helleseeth, 1976). *If F is of characteristic p with $[F : \mathbb{F}_p]$ a power of 2, then $W_{F,d}$ is not three-valued.*

Many attempts have been made to test or prove this conjecture in the case where F is of characteristic 2, and many fruitful discoveries were made in the process [12, 3, 20, 6, 2, 11]. The $p = 2$ case was at last proved in [15, Corollary 1.10]. Now the $p = 3$ case is proved in this paper as a corollary of a new bound on the p -divisibility of Weil sums.

For a nonzero integer n , the p -adic valuation of n , written $v_p(n)$, is the largest k such that $p^k \mid n$, and $v_p(0) = \infty$. If we extend this valuation to the cyclotomic field $\mathbb{Q}(e^{2\pi i/p})$ where the Weil sums lie, we may state the first fundamental result on the p -divisibility of Weil sums from [13, Theorem 4.5].

Theorem 1.3 (Helleseeth, 1976). *If F is of characteristic p , then we have $v_p(W_{F,d}(u)) > 0$ for every $u \in F$.*

It was recently proved [15, Theorems 1.7, 1.9] that when $W_{F,d}$ is three-valued, the values must be rational integers.

Theorem 1.4 (Katz, 2012). *If F is of characteristic p , and if $W_{F,d}$ is three-valued, then the three values are in \mathbb{Z} , one of the values is 0, and $d \equiv 1 \pmod{p-1}$.*

So if $W_{F,d}$ is three-valued, these two theorems show that $p \mid W_{F,d}(u)$ for all $u \in F$. Our main result is a much stronger lower bound on the p -divisibility.

Theorem 1.5. *If F is of characteristic p and order $q = p^n$, and if $W_{F,d}$ is three-valued with values 0, a , and b , then one of the following holds:*

- (i). $v_p(a), v_p(b) > n/2$; or
 - (ii). $v_p(a) = v_p(b) = n/2$, and $|a - b|$ is a power of p with $|a - b| > \sqrt{q}$;
- and case (ii) cannot occur if $p = 2$ or 3.

For the fields of interest in Conjecture 1.2, we use the techniques of [1] to show an upper bound on the p -divisibility of some $W_{F,d}(u)$.

Theorem 1.6. *Let F be of characteristic p and order $q = p^n$, with n a power of 2. If $W_{F,d}$ is three-valued, then there is some $u \in F^*$ such that $v_p(W_{F,d}(u)) \leq n/2$.*

This generalizes the result of Calderbank, McGuire, Poonen, and Rubinstein, who proved the $p = 2$ case in [3]. Theorems 1.5 and 1.6 immediately combine to prove Conjecture 1.2 in characteristic 2 and 3.

Theorem 1.7. *If F is of characteristic $p = 2$ or 3 with $[F : \mathbb{F}_p]$ a power of 2, then $W_{F,d}$ is not three-valued.*

In Section 2, we outline a group algebra approach to this problem inspired by the work of Feng [11], and in Section 3 we use a group-theoretic approach of McGuire [20] to determine congruences on the zero counts of critical polynomials that arise in our proofs. We then prove Theorem 1.5 in Section 4, and prove Theorem 1.6 in Section 5.

2. THE GROUP ALGEBRA AND THE FOURIER TRANSFORM

This section generalizes the group ring techniques of Feng [11] to arbitrary characteristic. As in the Introduction, F is a finite field.

Consider the group algebra $\mathbb{C}[F^*]$ over \mathbb{C} , whose elements are written as formal sums $S = \sum_{u \in F^*} S_u[u]$ with $S_u \in \mathbb{C}$. We identify any subset U of F^* with $\sum_{u \in U} [u]$ in $\mathbb{C}[F^*]$. For example, F^* itself is identified with $\sum_{u \in F^*} [u]$. If $S \in \mathbb{C}[F^*]$, we let $|S| = \sum_{u \in F^*} S_u$; if S represents a subset of F^* , this is indeed the cardinality of that set. Note that $SF^* = |S|F^*$ for any $S \in \mathbb{C}[F^*]$.

For $t \in \mathbb{Z}$ and $S = \sum_{u \in F^*} S_u[u] \in \mathbb{C}[F^*]$, we write $S^{(t)}$ to denote $\sum_{u \in F^*} S_u[u^t]$. Note that $|S^{(t)}| = |S|$. If $S \in \mathbb{C}[F^*]$, its conjugate is defined to be $\overline{S} = \sum_{u \in F^*} \overline{S_u}[u^{-1}]$, and note that $|\overline{S}| = \overline{|S|}$.

We let $\widehat{F^*}$ denote the group of multiplicative characters from F^* to \mathbb{C} . If $S = \sum_{u \in F^*} S_u[u] \in \mathbb{C}[F^*]$, and $\chi \in \widehat{F^*}$, we define $\chi(S) = \sum_{u \in F^*} S_u \chi(u)$. We call $\chi(S)$ the *Fourier coefficient of S at χ* , and we define the *Fourier transform of S* , denoted \widehat{S} , to be a function from $\widehat{F^*}$ to \mathbb{C} , where the value of \widehat{S} at χ is $\widehat{S}(\chi) = \chi(S)$.

It is straightforward to show that the Fourier transform $S \mapsto \widehat{S}$ is an isomorphism of \mathbb{C} -algebras from $\mathbb{C}[F^*]$ with its convolutional multiplication to the \mathbb{C} -algebra $\mathbb{C}^{\widehat{F^*}}$ of functions from $\widehat{F^*}$ into \mathbb{C} , with pointwise multiplication. This affords an inverse Fourier transform,

$$(3) \quad S_u = \frac{1}{|F^*|} \sum_{\chi \in \widehat{F^*}} \chi(S) \overline{\chi(u)},$$

so that $S = T$ if and only if $\chi(S) = \chi(T)$ for all $\chi \in \widehat{F^*}$.

Note that if t is an integer, then $\chi(S^{(t)}) = \chi^t(S)$. Similarly, $\chi(\overline{S}) = \overline{\chi(S)}$. If χ_0 is the principal character, then $\chi_0(S) = |S|$ for all $S \in \mathbb{C}[F^*]$.

As in the Introduction, we let d be a positive integer with $\gcd(d, |F^*|) = 1$, let $\psi: F \rightarrow \mathbb{C}$ be the canonical additive character, and we set

$$W_u = W_{F,d}(u) = \sum_{x \in F} \psi(x^d + ux).$$

We are interested in the Fourier analysis of the group algebra element

$$(4) \quad W = \sum_{u \in F^*} W_u[u],$$

which records the various values of $W_{F,d}$ as its coefficients. To this end, we introduce the group algebra element

$$\Psi = \sum_{u \in F^*} \psi(u)[u],$$

which has a close connection to W .

Lemma 2.1. $W = \Psi\Psi^{(-1/d)} + F^*$, where $1/d$ denotes the multiplicative inverse of d modulo $|F^*|$.

Proof. Note that $\Psi\Psi^{(-1/d)} = \sum_{y,z \in F^*} \psi(y)\psi(z)[yz^{-1/d}]$, but then reparameterize with $z = x^d$ and $y = ux$ to get $\Psi\Psi^{(-1/d)} = \sum_{u,x \in F^*} \psi(ux)\psi(x^d)[u] = \sum_{u \in F^*} (W_u - 1)[u]$, so that $\Psi\Psi^{(-1/d)} + F^* = W$. \square

Now we may carry out the Fourier analysis of Ψ and W .

Lemma 2.2. If $\chi \in \widehat{F^*}$ is not the principal character χ_0 , then $|\chi(\Psi)| = \sqrt{|F|}$, whereas $\chi_0(\Psi) = |\Psi| = -1$.

Proof. For any $\chi \in \widehat{F^*}$, we have

$$\chi(\Psi) = \sum_{u \in F^*} \psi(u)\chi(u),$$

which is a Gauss sum, of which the magnitude in general and the value when $\chi = \chi_0$ are well known (see [19, Theorem 5.11]). \square

Corollary 2.3. If $t \in \mathbb{Z}$ with $\gcd(t, |F^*|) = 1$, then $\Psi^{(t)}\overline{\Psi^{(t)}} = |F| - F^*$.

Proof. For $\chi \in \widehat{F^*}$, we have $\chi(\Psi^{(t)}\overline{\Psi^{(t)}}) = |\chi(\Psi^{(t)})|^2 = |\chi^t(\Psi)|^2$, which by Lemma 2.2 and the invertibility of t modulo $|F^*| = |\widehat{F^*}|$, equals 1 if χ is principal, and equals $|F|$ otherwise. One obtains the same values by applying χ to $|F| - F^*$, so that $\Psi^{(t)}\overline{\Psi^{(t)}} = |F| - F^*$ by (3). \square

Corollary 2.4. $W\overline{W} = |F|^2$.

Proof. In view of Lemma 2.1, we multiply $W = \Psi\Psi^{(-1/d)} + F^*$ by its conjugate. Note that $\overline{F^*} = F^*$, recall that $SF^* = |S|F^*$ for any $S \in \mathbb{C}[F^*]$, and use Lemma 2.2 to see that $|\Psi^{(-1/d)}| = |\Psi| = -1$ and thus $|\overline{\Psi^{(-1/d)}}| =$

$|\overline{\Psi}| = -1$, so that $W\overline{W} = \Psi\overline{\Psi}\Psi^{(-1/d)}\overline{\Psi^{(-1/d)}} + (|F^*| + 2)F^*$. Then apply Corollary 2.3 to the first term to finish. \square

Now define

$$(5) \quad X = \sum_{u \in F^*} W_u^2[u]$$

and

$$(6) \quad V = \sum_{v \in F} [(v^d + (1-v)^d)^{1/d}],$$

where the $1/d$ is understood modulo $|F^*|$. We claim that $v^d + (1-v)^d \neq 0$ for any $v \in F$, because $v \neq v-1$, and the condition $\gcd(d, |F^*|) = 1$ makes $x \mapsto x^d$ a permutation of F and makes d odd when the characteristic of F is odd. This makes V a legitimate element of $\mathbb{C}[F^*]$ with

$$(7) \quad |V| = |F|.$$

We can now relate W , X , and V .

Lemma 2.5. $X = WV$.

Proof. Note that

$$\begin{aligned} (WV)_u &= \sum_{z \in F^*} W_{u/z} V_z \\ &= \sum_{v \in F} W_{u(v^d + (1-v)^d)^{-1/d}} \\ &= \sum_{v, w \in F} \psi(w^d + u(v^d + (1-v)^d)^{-1/d} w) \\ &= |F| + \sum_{v \in F} \sum_{w \in F^*} \psi(w^d + u(v^d + (1-v)^d)^{-1/d} w). \end{aligned}$$

Since $\gcd(d, |F^*|) = 1$ makes d odd when F is of odd characteristic, the map $(x, y) \mapsto (x/(x+y), (x^d + y^d)^{1/d})$ is a bijection from $\{(x, y) \in F^2 : x+y \neq 0\}$ to $F \times F^*$, with inverse $(v, w) \mapsto (v^d + (1-v)^d)^{-1/d}(vw, (1-v)w)$. We may then reparameterize our sum to obtain

$$\begin{aligned} (WV)_u &= |F| + \sum_{\substack{x, y \in F \\ x+y \neq 0}} \psi(x^d + y^d + u(x+y)) \\ &= \sum_{x, y \in F} \psi(x^d + y^d + u(x+y)) \\ &= W_u^2. \end{aligned} \quad \square$$

It is now easy to calculate the first four power moments of the Weil sum.

Corollary 2.6. *We have*

$$(i). \quad \sum_{u \in F^*} W_u = |F|,$$

- (ii). $\sum_{u \in F^*} W_u^2 = |F|^2$,
- (iii). $\sum_{u \in F^*} W_u^3 = |F|^2 V_1$, and
- (iv). $\sum_{u \in F^*} W_u^4 = |F|^2 \sum_{u \in F^*} V_u^2$.

Proof. First of all, $\sum_{u \in F^*} W_u = |W|$, which equals $|\Psi| |\Psi^{(-1/d)}| + |F^*| = (-1)^2 + |F^*|$ by Lemmata 2.1 and 2.2.

Secondly, $\sum_{u \in F^*} W_u^2 = |X|$, which equals $|W| \cdot |V| = |F|^2$ by Lemma 2.5, equation (7), and the previous part.

Thirdly, $\sum_{u \in F^*} W_u^3$ is the coefficient for the element $[1]$ in $X\overline{W}$, and $X\overline{W} = (WV)\overline{W} = |F|^2 V$ by Lemma 2.5 and Corollary 2.4.

Finally, $\sum_{u \in F^*} W_u^4$ is the coefficient of $[1]$ in $X\overline{X}$, and we see that $X\overline{X} = (WV)(\overline{WV}) = |F|^2 V\overline{V}$ by Lemma 2.5 and Corollary 2.4. \square

3. CONGRUENCES FOR V_1

As in previous sections, F is a finite field and d is a positive integer with $\gcd(d, |F^*|) = 1$. We recall V from equation (6), whose coefficient V_1 appears in Corollary 2.6.iii. We deduce useful congruences for V_1 by means of a group action studied by McGuire [20].

Proposition 3.1. *Let $V_1 = |\{v \in F : v^d + (1-v)^d = 1\}|$.*

- (i). *If $|F| \equiv 0 \pmod{3}$, then $V_1 \equiv 3 \pmod{6}$.*
- (ii). *If $|F| \equiv 1 \pmod{3}$, then*
 - (a). *if $2^{d-1} = 1$ in F , then $V_1 \equiv 1 \pmod{6}$, but*
 - (b). *if $2^{d-1} \neq 1$ in F , then $V_1 \equiv 4 \pmod{6}$.*
- (iii). *If $|F| \equiv 2 \pmod{3}$, then*
 - (a). *if $2^{d-1} = 1$ in F , then $V_1 \equiv 5 \pmod{6}$, but*
 - (b). *if $2^{d-1} \neq 1$ in F , then $V_1 \equiv 2 \pmod{6}$.*

Before proving the proposition, we interject two remarks.

Remark 3.2. When F is of order 2^n , then we have $2 = 0$ in F , so that $2^{d-1} \neq 1$, and we recover the result of McGuire (cf. [20, Corollary 1]) that $V_1 \equiv 4 \pmod{6}$ if n is even, and $V_1 \equiv 2 \pmod{6}$ if n is odd.

Remark 3.3. When we are dealing with three-valued Weil sums, Theorem 1.4 shows that $d \equiv 1 \pmod{p-1}$, in which case $2^{d-1} = 1$ in F by Fermat's Little Theorem whenever $p \neq 2$.

Proof of Proposition 3.1. We are counting modulo 6 the roots of $f(X) = X^d + (1-X)^d - 1$ in F . First of all, note that 0 and 1 are always roots of f . Consider the action of the two involutions $\sigma(x) = 1-x$ and $\tau(x) = 1/x$ on $F \setminus \{0, 1\}$. Note that $f(\sigma(x)) = f(x)$ and $f(\tau(x)) = -f(x)/x^d$, because $\gcd(d, |F^*|) = 1$ makes d odd when $|F|$ is odd. Thus σ and τ map the roots of f in $F \setminus \{0, 1\}$ to themselves.

For generic F , our σ and τ generate a group G isomorphic to S_3 , consisting of the identity, σ , τ , $\tau \circ \sigma(x) = 1/(1-x)$, $\sigma \circ \tau(x) = (x-1)/x$, and

$\sigma \circ \tau \circ \sigma(x) = \tau \circ \sigma \circ \tau(x) = x/(x-1)$. The set of roots of f in $F \setminus \{0, 1\}$ is preserved by G , so is partitioned into orbits under its action.

A G -orbit contains 6 elements unless it contains a point fixed by some nonidentity element of G . An $x \in F \setminus \{0, 1\}$ is fixed by a nonidentity element of G if and only if it satisfies at least one of the following equations: $x = 1-x$, $x = 1/x$, $x = 1/(1-x)$, $x = (x-1)/x$, or $x = x/(x-1)$. Equivalently, x equals at least one of: (i) -1 , (ii) 2 , (iii) the multiplicative inverse of 2 (which we shall call $1/2$ when it exists), or (iv) a root of $\Phi_6(X) = X^2 - X + 1$, the cyclotomic polynomial of index 6.

If $|F| \equiv 0 \pmod{3}$, then $-1 = 2 = 1/2$, which is the double root of Φ_6 . Furthermore, $f(-1) = 0$ since d is odd. Thus the roots of f consist of $0, 1, -1$, and orbits of size 6, so that $V_1 \equiv 3 \pmod{6}$.

If F is of characteristic 2, then (i) $-1 = 1$, (ii) $2 = 0$, and (iii) the multiplicative inverse of 2 does not exist, so these are not points in $F \setminus \{0, 1\}$, and (iv) the roots of Φ_6 are the primitive third roots of unity, ω and $\omega^{-1} = 1 - \omega$, which lie in F if and only if $|F| \equiv 1 \pmod{3}$. In this case $f(\omega) = f(\omega^{-1}) = 0$ since $d \equiv \pm 1 \pmod{3}$, because $\gcd(d, |F^*|) = 1$. So the roots of f consist of 0 and 1 always, the third roots of unity if $|F| \equiv 1 \pmod{3}$, and orbits of size 6. Thus $V_1 \equiv 4 \pmod{6}$ if $|F| \equiv 1 \pmod{3}$ and $V_1 \equiv 2 \pmod{6}$ if $|F| \equiv 2 \pmod{3}$.

Now suppose that F is of characteristic $p \geq 5$. Then (i) -1 , (ii) 2 , (iii) $1/2$, and (iv) the roots of Φ_6 are five distinct elements in characteristic p . The roots ζ and $\zeta^{-1} = 1 - \zeta$ of Φ_6 lie in F if and only if $|F| \equiv 1 \pmod{6}$, or equivalently, if and only if $|F| \equiv 1 \pmod{3}$. In this case $f(\zeta) = f(\zeta^{-1}) = 0$ since $d \equiv \pm 1 \pmod{6}$ because $\gcd(d, |F^*|) = 1$, so the roots of Φ_6 are roots of f if they are present in F . The three elements $-1, 2$, and $1/2$ make up a G -orbit, and are roots of f if and only if $0 = f(1/2) = (1/2)^d + (1/2)^d - 1 = (1/2)^{d-1} - 1$, that is, if and only if $2^{d-1} = 1$. So the roots of f consist of 0 and 1 always, the roots of Φ_6 if and only if $|F| \equiv 1 \pmod{3}$, the elements $-1, 2$, and $1/2$ if and only if $2^{d-1} = 1$ in F , and orbits of size 6. Adding these counts together modulo 6 according to the four possible cases finishes the proof. \square

4. PROOF OF THEOREM 1.5

Throughout this section, we assume that F is a finite field of characteristic p and order q , and that d is a positive integer with $\gcd(d, q-1) = 1$. We let $\psi: F \rightarrow \mathbb{C}$ be the canonical additive character, and set

$$W_u = W_{F,d}(u) = \sum_{x \in F} \psi(x^d + ux),$$

and assume that $W_{F,d}$ is three-valued. Per Theorem 1.4, these three values must all be in \mathbb{Z} , and one of them must be 0. Call the other two values a and b ; these are of opposite sign since $\sum_{u \in F^*} W_u^2 = (\sum_{u \in F^*} W_u)^2$ by Corollary 2.6.i-ii. We calculate the power moments in this three-valued case.

Lemma 4.1. *For any positive integer k , we have*

$$\sum_{u \in F^*} W_u^k = \frac{q^2(a^{k-1} - b^{k-1}) - qab(a^{k-2} - b^{k-2})}{a - b}.$$

Proof. The $k = 1$ and 2 cases are proved in Corollary 2.6.i–ii. For $k > 2$, we note that $W_u^{k-2}(W_u - a)(W_u - b) = 0$ for all $u \in F^*$, whence

$$\sum_{u \in F^*} W_u^k = (a + b) \sum_{u \in F^*} W_u^{k-1} - ab \sum_{u \in F^*} W_u^{k-2},$$

from which the identity for $\sum_{u \in F^*} W_u^k$ follows by induction. \square

We let V be as defined in equation (6), and note the consequences that our power moment results have for the coefficients of V .

Lemma 4.2. $V_1 = a + b - \frac{ab}{q}$, hence $v_p(ab) \geq v_p(q)$, with strict inequality if $p = 2$ or 3 .

Proof. Compare Corollary 2.6.iii and the $k = 3$ case of Lemma 4.1 to deduce the equation. Since $V_1 = |\{v \in F : v^d + (1 - v)^d = 1\}|$, and since $a, b \in \mathbb{Z}$, we know that ab/q lies in \mathbb{Z} , so that $v_p(ab) \geq v_p(q)$. Suppose that F is of characteristic $p = 2$ or 3 . Since $W_{F,d}$ is three-valued, $d \equiv 1 \pmod{p-1}$ by Theorem 1.4, and so by Proposition 3.1.i and Remark 3.2, we see that $V_1 \equiv 0 \pmod{p}$. Furthermore, $a \equiv b \equiv 0 \pmod{p}$ by Theorem 1.3. Thus $ab/q \equiv 0 \pmod{p}$, so $v_p(ab) > v_p(q)$. \square

Lemma 4.3. *We have*

- (i). $\sum_{u \in \mathbb{F} \setminus \{0,1\}} V_u = \frac{(q-a)(q-b)}{q} > 0$, and
- (ii). $\sum_{u \in \mathbb{F} \setminus \{0,1\}} V_u^2 = -\frac{ab(q-a)(q-b)}{q^2} > 0$.

Proof. For part (i), we have $\sum_{u \in \mathbb{F} \setminus \{0,1\}} V_u = |V| - V_1$, and then use equation (7) and the value of V_1 from Lemma 4.2. For part (ii), compare Corollary 2.6.iv and the $k = 4$ case of Lemma 4.1 to see that

$$q^2 \sum_{u \in F^*} V_u^2 = q^2(a^2 + ab + b^2) - qab(a + b),$$

and then substitute the value of V_1 from Lemma 4.2 and rearrange. Both our sums are strictly positive since $|a|, |b| < q$, inasmuch as Corollary 2.6.ii shows that $q^2 = \sum_{u \in F^*} W_u^2$, which must be at least $a^2 + b^2$. \square

Now write $|a| = a_o a_p$, $|b| = b_o b_p$, and $|a - b| = (a - b)_o (a - b)_p$, where a_p , b_p , and $(a - b)_p$ are powers of p and a_o , b_o , and $(a - b)_o$ are positive integers coprime to p . So $\gcd(a_o, b_o)$ is coprime to p , and yet also a divisor of every W_u , hence of $|W| = q$ (see Corollary 2.6.i), and so $\gcd(a_o, b_o) = 1$. Thus a_o , b_o , and $(a - b)_o$ are pairwise coprime, and now we show that they divide most of the coefficients of V .

Lemma 4.4. $a_o b_o (a - b)_o \mid V_u$ for all $u \in F^*$ with $u \neq 1$.

Proof. Recall the definitions of W and X in equations (4) and (5). Let $A = \{u \in F^* : W_u = a\}$ and $B = \{u \in F^* : W_u = b\}$, so that $W = aA + bB$ and $X = a^2A + b^2B$. Since $X = WV$ by Lemma 2.5, we solve for A to get $a(a-b)A = W(V-b)$, then multiply both sides by \overline{W} and apply Corollary 2.4 to get $a(a-b)A\overline{W} = q^2(V-b)$. Since A and \overline{W} have integer coefficients, $a_o(a-b)_o$ divides all the coefficients of $V-b$, in particular, all V_u with $u \neq 1$. By the same method, we deduce that $b_o(a-b)_o$ divides all V_u with $u \neq 1$, and recall that a_o , b_o , and $(a-b)_o$ are pairwise coprime. \square

Now we recall and prove our main result, Theorem 1.5.

Theorem 4.5. *If $q = p^n$, then one of the following holds:*

- (i). $v_p(a), v_p(b) > n/2$; or
 - (ii). $v_p(a) = v_p(b) = n/2$, and $|a-b|$ is a power of p with $|a-b| > \sqrt{q}$;
- and case (ii) cannot occur if $p = 2$ or 3 .

Proof. By Lemma 4.4, $V_u^2 \geq a_ob_o(a-b)_o V_u$ for each $u \neq 0, 1$. Thus

$$\sum_{u \in \mathbb{F} \setminus \{0,1\}} V_u^2 \geq a_ob_o(a-b)_o \sum_{u \in \mathbb{F} \setminus \{0,1\}} V_u,$$

so by Lemma 4.3 we can divide by the sum on the right to obtain $-ab/q \geq a_ob_o(a-b)_o$, which yields

$$(8) \quad a_p b_p \geq q(a-b)_o.$$

If $a_p \neq b_p$, set $\{g, h\} = \{a_p, b_p\}$ with $g < h$. Then $(a-b)_p = g$, so $(a-b)_o = |a-b|/g$, and thus (8) becomes $g^2 h \geq q|a-b|$. Now $|a-b| > \max\{|a|, |b|\} \geq h$, so that $g^2 > q$, and so $a_p, b_p > \sqrt{q}$, and hence $v_p(a), v_p(b) > n/2$.

If $a_p = b_p$, then $v_p(ab) \geq v_p(q)$ by Lemma 4.2. If this inequality is strict, which it must be if $p = 2$ or 3 , then $v_p(a) = v_p(b) > n/2$.

So it remains to consider the case where $p \geq 5$ and $v_p(a) = v_p(b) = n/2$. Then $a_p b_p = q$, so (8) forces $(a-b)_0 = 1$, hence $|a-b|$ must be a power of p , and indeed must be greater than \sqrt{q} since $|a-b| > |a|$ and a is a nonzero integral multiple of \sqrt{q} . \square

We conclude with a remark about what happens when we are in case (ii) of our theorem, and what that tells us about possible counterexamples to Conjecture 1.2.

Remark 4.6. In case (ii) of our theorem, where $v_p(a) = v_p(b) = n/2$, we have $-ab/q = a_ob_o$, so that $V_1 = a + b + a_ob_o$ by Lemma 4.2, and $\sum_{v \in F \setminus \{0,1\}} V_u(V_u - a_ob_o) = 0$ by Lemma 4.3. Since V_u is equal to the count $|\{v \in F : v^d + (1-v)^d = u^d\}|$, and since Lemma 4.4 tells us that $a_ob_o \mid V_u$ for all $u \in F \setminus \{0,1\}$, we see that $V_u \in \{0, a_ob_o\}$ for all $u \in F \setminus \{0,1\}$. Furthermore, $|a-b|$ is a power of p greater than \sqrt{q} , so that $a_o + b_o$ is also a power of p , and $a_o + b_o \geq p \geq 5$, and so $a_ob_o \geq 4$.

Theorem 1.6 forces us into this $v_p(a) = v_p(b) = n/2$ case when n is a power of 2. So if there is a counterexample to Conjecture 1.2, the field F

and exponent d must have the property that $v^d + (1 - v)^d$ represents 1 for precisely $a + b + a_o b_o$ values of $v \in F$, and it represents $\frac{g-a-b-a_o b_o}{a_o b_o}$ elements of $F \setminus \{0, 1\}$ for precisely $a_o b_o$ values of $v \in F$ each, and it does not represent any other element of F .

5. PROOF OF THEOREM 1.6

Throughout this section, we use the definition of $W_{F,d}(u)$ from (1). We prove Theorem 1.6 using two results of Aubry, Katz, and Langevin [1]. The first key result is Corollary 4.2 of that paper.

Proposition 5.1 (Aubry-Katz-Langevin, 2013). *Let K be a finite field and L an extension of K of finite degree, and suppose that d is a positive integer with $\gcd(d, |L^*|) = 1$. Then*

$$\min_{u \in L^*} v_p(W_{L,d}(u)) \leq [L : K] \cdot \min_{u \in K^*} v_p(W_{K,d}(u)).$$

The second result is part of Corollary 4.4 of the same paper.

Proposition 5.2 (Aubry-Katz-Langevin, 2013). *Let K be a finite field of characteristic p , and let L be the quadratic extension of K . Let d be a positive integer with $\gcd(d, |L^*|) = 1$, and suppose that d is degenerate over K but not over L . Then*

$$\min_{u \in L^*} v_p(W_{L,d}(u)) = [K : \mathbb{F}_p].$$

We now recall and prove Theorem 1.6.

Theorem 5.3. *Let F be a field of characteristic p and order $q = p^n$, with n a power of 2. Let d be a positive integer with $\gcd(d, q-1) = 1$ such that $W_{F,d}$ is three-valued. Then there is some $u \in F^*$ such that $v_p(W_{F,d}(u)) \leq n/2$.*

Proof. By Theorem 1.4, d is degenerate over \mathbb{F}_p . But d is not degenerate over F since $W_{F,d}$ is three-valued (cf. (2)). Proceeding by successive quadratic extensions from \mathbb{F}_p to F , there must be subfields, say K and L , of F with $[L : K] = 2$ and d degenerate over K but not L . Then by Propositions 5.1 and 5.2, we have

$$\begin{aligned} \min_{u \in F^*} v_p(W_{F,d}(u)) &\leq [F : L] \cdot \min_{u \in L^*} v_p(W_{L,d}(u)) \\ &= [F : L][K : \mathbb{F}_p], \end{aligned}$$

so that there is some $u \in F^*$ with $v_p(W_{F,d}(u)) \leq [F : \mathbb{F}_p]/[L : K] = n/2$. \square

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